

# GORENSTEIN INJECTIVE DIMENSION AND A GENERALIZATION OF ISCHEBECK FORMULA

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**ABSTRACT.** Let  $(R, \mathfrak{m})$  be a commutative Noetherian local ring and let  $M$  and  $N$  be finitely generated  $R$ -modules of finite injective dimension and finite Gorenstein injective dimension, respectively. In this paper we prove a generalization of Ischebeck Formula, that is  $\text{depth}_R M + \sup\{i \mid \text{Ext}_R^i(M, N) \neq 0\} = \text{depth} R$ .

## 1. INTRODUCTION

Throughout this paper,  $(R, \mathfrak{m})$  is a commutative Noetherian local ring with the maximal ideal  $\mathfrak{m}$ .

Our motivation to do this work is the first steps of the solution of a conjecture of Bass given by Levin and Vasconcelos in 1968 [LV] when  $R$  admits a finitely generated  $R$ -module of injective dimension  $\leq 1$ . One of the main ingredients of their proof is to use a formula which relates the depth of a module to the depth of the ring via a non-vanishing of  $\text{Ext}$  when  $R$  admits a finitely generated  $R$ -module of finite injective dimension. More precisely, if  $M$  is a finitely generated  $R$ -module of finite injective dimension, then for any finitely generated  $R$ -module  $L$ , there is the following equality

$$\text{depth}_R L + \sup\{i \mid \text{Ext}_R^i(L, M) \neq 0\} = \text{depth} R.$$

This formula was simultaneously proved by Ischebeck [I] using an easy induction argument on  $\text{depth}_R L$ . In this paper we give and prove a generalization of this formula existing a finitely generated  $R$ -module of finite Gorenstein injective dimension which turns out to be much more complicated. To be more precise, we show that if  $M$  and  $N$  are finitely generated  $R$ -modules of finite injective dimension and finite Gorenstein injective dimension respectively, then we have

$$\text{depth}_R M + \sup\{i \mid \text{Ext}_R^i(M, N) \neq 0\} = \text{depth} R.$$

As an application, we show that if  $R$  is Cohen-Macaulay and  $N$  is a finitely generated  $R$ -module of finite Gorenstein injective dimension, then there exists a finitely generated  $R$ -module  $M$  of finite injective dimension such that  $\text{Hom}_R(M, N) \neq 0$  and  $\text{Ext}_R^i(M, N) = 0$  for all  $i > 0$ .

## 2. THE MAIN RESULTS

Throughout this section, let  $(R, \mathfrak{m})$  be a local ring and we let  $\hat{R}$ , the completion of  $R$  with respect to the maximal ideal  $\mathfrak{m}$ . For an  $R$ -module  $M$ , a prime ideal  $\mathfrak{p}$  of  $R$  and an integer  $i$ ,

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2000 *Mathematics Subject Classification.* 13D05, 13D07, 13H10.

*Key words and phrases.* Gorenstein injective, mock finitely generated and Cohen-Macaulay ring.

the  $i$ -th Bass number of  $M$  with respect to  $\mathfrak{p}$  is the (cardinal) number  $\mu_R^i(\mathfrak{p}, M)$ , that is the dimension of  $\text{Ext}_{R_{\mathfrak{p}}}^i(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, M_{\mathfrak{p}})$  as a vector-space on the residue field  $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ .

**Lemma 2.1.** *Let  $N$  be a finitely generated  $R$ -module of finite Gorenstein injective dimension and let  $M$  be a finitely generated  $R$ -module of finite injective dimension with  $\text{depth}_R M = \text{depth}_R R = r > 0$ . Then we have the following equality  $\sup\{i \mid \text{Ext}_R^i(M, N) \neq 0\} = 0$ .*

*Proof.* We note that  $\text{id}_R M = \text{id}_{\hat{R}}(M \otimes_R \hat{R}) = r$  and by virtue of [FFr, Theorem 3.6] we have  $\text{Gid}_{\hat{R}}(N \otimes_R \hat{R}) < \infty$ . Therefore it follows from [CS, Corollary 2.3] that  $\text{Gid}_R N = \text{Gid}_{\hat{R}}(N \otimes_R \hat{R}) = r$ , and so without loss of generality the claim, we may assume that  $R$  is complete. Since  $M$  is finitely generated and  $\text{depth}_R M = r$ , the proof of [FFGR, Proposition 2.2] and [PS, Theorem 4.10] imply that for each  $i \geq 0$ , there is the following isomorphisms

$$\text{Ext}_R^i(M, N) \cong \text{Ext}_R^{i+r}(E^r, N) \cong \oplus_{\mu_R^r(\mathfrak{m}, M)} \text{Ext}_R^{i+r}(E(R/\mathfrak{m}), N).$$

Now, using [EJ2, Corollary 4.4], we have the following equalities which gives the assertion

$$r = \text{Gid}_R N = \sup\{i \mid \text{Ext}_R^i(E(R/\mathfrak{m}), N) \neq 0\}.$$

□

**Definition 2.2.** Let  $N$  be an  $R$ -module. A complex  $\dots \rightarrow E_1 \rightarrow E_0 \rightarrow N \rightarrow 0$  is called an *injective resolvent* of  $N$  if for each injective  $R$ -module  $E$ , the functor  $\text{Hom}_R(E, -)$  leaves it exact. Let  $M$  and  $N$  be two  $R$ -modules and let  $\dots \rightarrow E_1 \rightarrow E_0 \rightarrow N \rightarrow 0$  and  $0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \dots$  be injective resolvent and injective resolution for the corresponding modules  $N$  and  $M$ . Then the 3rd quadrant double complex  $(\text{Hom}(E^i, E_j))_{i,j}$  is such that the two associated spectral sequences collapse. This implies that we can compute derived functors of  $\text{Hom}$  using either the injective resolution of  $M$  or the injective resolvent of  $N$  when it exists. These left derived functors will be denoted  $\text{Ext}_n^R(M, N)$ . From the definition of  $\text{Ext}_0^R(M, N)$  it is clear that there is a natural transformation  $\text{Ext}_0^R(M, N) \rightarrow \text{Ext}_R^0(M, N) = \text{Hom}_R(M, N)$ . The image of  $\text{Ext}_0^R(M, N)$  in  $\text{Hom}_R(M, N)$  consists of those linear maps  $M \rightarrow N$  which can be factored through an injective  $R$ -module. We will let  $\overline{\text{Ext}}_0^R(M, N)$  and  $\overline{\text{Ext}}_R^0(M, N)$  denote the kernel and cokernel of the natural transformation  $\text{Ext}_0^R(M, N) \rightarrow \text{Hom}_R(M, N)$ . Following [EJ1], an  $R$ -module  $N$  is said to be *mock finitely generated* if for each finitely generated  $R$ -module  $M$  and each  $i \geq 1$  each of the modules  $\text{Ext}_R^i(M, N)$ ,  $\text{Ext}_i^R(M, N)$ ,  $\overline{\text{Ext}}_0^R(M, N)$  and  $\overline{\text{Ext}}_R^0(M, N)$  is finitely generated.

**Remark 2.3.** Foxby in [F, Corollary 4.5] proved that if  $M$  is a non-zero finitely generated  $R$ -module with  $t = \text{depth}_R M < \dim R$ , then  $\mu_R^{t+1}(\mathfrak{m}, M) > \mu_R^t(\mathfrak{m}, M) > 0$ . In the following lemma which we use from this fact,  $M$  is of finite injective dimension and so  $R$  is Cohen-Macaulay. In this case the result has an easy proof which we mention it. In fact, without loss of generality we may assume that  $R$  is complete and so  $R$  is a homomorphic image of a Gorenstein local ring  $(A, \mathfrak{n})$  and we have  $\text{depth}_R M = \text{depth}_A M$ . By virtue of [F, Remark 2.7] there exists a complex of finitely generated free  $R$ -modules  $\mathcal{F} := \dots \rightarrow F_{t+1} \rightarrow F_t \rightarrow 0$  such that the rank of each  $F_i$  is  $\mu_R^i(\mathfrak{m}, M)$  and  $H_i(\mathcal{F}) = 0$  for all  $i < \text{depth}_R M$  or  $i > \dim M$  and moreover  $\dim H_i(\mathcal{F}) \leq i$  for all

*i.* By using the local duality theorem for local cohomology and this fact that  $H_n^t(M) \neq 0$  we can deduce that  $H_t(\mathcal{F}) \neq 0$ . Now consider the exact sequence  $0 \rightarrow Z \rightarrow F_{t+1} \rightarrow F_t \rightarrow H_t(\mathcal{F}) \rightarrow 0$ . As  $R$  is Cohen-Macaulay, we have  $\dim Z = \dim R$  and since  $t < \dim R$ , there exists a  $\mathfrak{p} \in \text{Ass} Z$  such that  $\dim R/\mathfrak{p} > t$  and so  $H_t(\mathcal{F})_{\mathfrak{p}} = 0$ . Now if we localize the above exact sequence to  $\mathfrak{p}$  we deduce that  $\text{rank} F_{t+1} > \text{rank} F_t$ . Keeping in mind this fact we have the following lemma.

**Lemma 2.4.** *Let  $N$  be a finitely generated  $R$ -module of finite Gorenstein injective dimension and let  $M$  be a finitely generated  $R$ -module of finite injective dimension with  $\text{depth} M = 0$  and  $\text{depth} R = 1$ , then we have the equality  $\sup\{i \mid \text{Ext}_R^i(M, N) \neq 0\} = 1$ .*

*Proof.* Similar to the argument mentioned in the previous lemma we may assume that  $R$  is complete. As  $\text{Gid}_R N = \text{depth} R = 1$ , it is clear that  $\text{Ext}_R^i(M, N) = 0$  for all  $i > 1$  and the same reasoning mentioned in Lemma 2.1 and [EJ2, Corollary 4.4] imply that  $\text{Ext}_R^1(E(R/\mathfrak{m}), N) \neq 0$  and  $\text{Ext}_R^1(E(R/\mathfrak{p}), N) = 0$  for all  $\mathfrak{p} \in \text{Spec} R \setminus \{\mathfrak{m}\}$ . Now, it remains to show that  $\text{Ext}_R^1(M, N) \neq 0$ . Assume that  $\text{Ext}_R^1(M, N) = 0$ . As  $\text{Gid}_R N = \text{id}_R M = 1$ , there exists an exact sequence of  $R$ -modules  $0 \rightarrow N \rightarrow G \rightarrow K \rightarrow 0$  such that  $G$  is Gorenstein injective and  $K$  is injective and moreover there is a minimal injective resolution for  $M$  as  $0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow 0$ . we note that since  $\text{depth} M = 0$ , the injective module  $E(R/\mathfrak{m})$  appears in each  $E^i$ . Now in view of [EJ1, Proposition 2.4], we have the following commutative diagram with the rows and columns exact

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \parallel & & \parallel \\
 & & \overline{\text{Ext}}_0^R(M, N) & \longrightarrow & \overline{\text{Ext}}_0^R(M, G) & \longrightarrow & \overline{\text{Ext}}_0^R(M, K) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 = \text{Ext}_1^R(M, K) & \longrightarrow & \text{Ext}_0^R(M, N) & \longrightarrow & \text{Ext}_0^R(M, G) & \longrightarrow & \text{Ext}_0^R(M, K) \longrightarrow 0 \\
 & & \downarrow \beta & & \downarrow \alpha & & \downarrow \delta \\
 0 & \longrightarrow & \text{Hom}_R(M, N) & \longrightarrow & \text{Hom}_R(M, G) & \longrightarrow & \text{Hom}_R(M, K) \longrightarrow \text{Ext}_R^1(M, N) = 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \overline{\text{Ext}}_R^0(M, N) & & 0 & & 0 \\
 & & \downarrow & & & & \\
 & & 0 & & & & 
 \end{array}$$

The Snake Lemma implies the following exact sequence of  $R$ -modules

$$0 = \text{Ker} \delta \rightarrow \text{Coker} \beta \cong \overline{\text{Ext}}_R^0(M, N) \rightarrow \text{Coker} \alpha = 0$$

which implies that  $\overline{\text{Ext}}_R^0(M, N) = 0$ . On the other hand, by virtue of [EJ1, Proposition 1.6] there is the following exact sequence

$$0 = \overline{\text{Ext}}_R^0(M, N) \rightarrow \text{Ext}_R^1(E^1, N) \rightarrow \text{Ext}_R^1(E^0, N) \rightarrow \text{Ext}_R^1(M, N) = 0$$

and so since  $M$  is finitely generated, we have the following isomorphisms

$$\oplus_{\mu_R^1(\mathfrak{m}, M)} \text{Ext}_R^1(E(R/\mathfrak{m}), N) \cong \text{Ext}_R^1(E^1, N) \cong \text{Ext}_R^1(E^0, N) \cong \oplus_{\mu_R^0(\mathfrak{m}, M)} \text{Ext}_R^1(E(R/\mathfrak{m}), N).$$

The preceding paragraph implies that  $\mu_R^1(\mathfrak{m}, M) = \mu_R^0(\mathfrak{m}, M)$ . But as  $\text{depth}_R M < \text{depth} R \leq \dim R$ , the last statement is in contradiction with Remark 2.3 and so  $\text{Ext}_R^1(M, N) \neq 0$ .  $\square$

**Theorem 2.5.** *Let  $M$  be a finitely generated  $R$ -module of finite injective dimension and let  $N$  be a finitely generated  $R$ -module of finite Gorenstein injective dimension. Then there is the following equality*

$$\text{depth}_R M + \sup\{i \mid \text{Ext}_R^i(M, N) \neq 0\} = \text{depth} R.$$

*Proof.* As all invariants in the assumption and the assertion are well-behaviour with respect to the completion of  $R$  in  $\mathfrak{m}$ , we may assume that  $R$  is complete. At first assume that  $\text{depth} R = 0$ . In this case  $\text{id}_R M = 0 = \text{Gid}_R N$ , and so  $M = \bigoplus_{\mu_R^0(\mathfrak{m}, M)} E(R/\mathfrak{m})$ . Therefore the result follows by the same reasoning in Lemma 2.1 and using [EJ2, Corollary 4.4] in this case. Assume that  $\text{depth} R = r > 0$  and  $\text{depth}_R M = s$  and we also note that  $s \leq \dim M \leq \text{id}_R M = r$ . If  $r = s$ , then the assertion follows by Lemma 2.1. Therefore we assume that  $s < r$ . In this case if  $r = 1$ , then the assertion follows by Lemma 2.4. Therefore we may assume that  $r \geq 2$ . As  $\text{Gid}_R N = r$ , there is an exact sequence of  $R$ -modules

$$0 \rightarrow N \rightarrow G \rightarrow K \rightarrow 0 \quad (\dagger)$$

such that  $G$  is mock finitely generated Gorenstein injective and  $K$  is mock finitely generated with  $\text{id}_R K = r - 1$  (see the proof of [S, Proposition 5.3]). We claim that  $\mu_R^{r-1}(\mathfrak{m}, K) > 0$ , otherwise there exists a prime ideal  $\mathfrak{p}$  contained in  $\mathfrak{m}$  such that  $\text{Ext}_R^{r-1}(R/\mathfrak{p}, K) \neq 0$ . Now, since  $\dim R/\mathfrak{p} = t > 0$ , a similar proof mentioned in [B, Lemma 3.1] implies that  $\mu_R^{r-1+t}(\mathfrak{m}, K) \neq 0$ , that is in contradiction with  $\text{id}_R K = r - 1$ . Now, we proceed the rest of proof by induction on  $s$ . If  $s = 0$ , there is a monomorphism  $0 \rightarrow R/\mathfrak{m} \rightarrow M$  and so there is an epimorphism  $\text{Ext}_R^{r-1}(M, K) \rightarrow \text{Ext}_R^{r-1}(R/\mathfrak{m}, K) \rightarrow 0$ . Since  $\text{Ext}_R^{r-1}(R/\mathfrak{m}, K) \neq 0$ , we have  $\text{Ext}_R^{r-1}(M, K) \neq 0$ . On the other hand it is clear that  $\text{Ext}_R^i(M, K) = 0$  for all  $i > r - 1$  as  $\text{id}_R K = r - 1$ . Now, if we apply the functor  $\text{Hom}_R(M, -)$  to that exact sequence  $(\dagger)$ , the assertion follows easily in this case. Suppose that  $s > 0$  and the assertion has been proved for all values smaller than  $s$  and so we prove it for  $s$ . In this case there exists an element  $x \in \mathfrak{m}$  which is an  $M$ -regular. Hence there exists the following exact sequence of  $R$ -modules

$$0 \rightarrow M \xrightarrow{x} M \rightarrow M/xM \rightarrow 0 \quad (\ddagger).$$

Applying the functor  $\text{Hom}_R(-, N)$  to the exact sequence  $(\ddagger)$ , for each  $i$ , induces the following exact sequence

$$\text{Ext}_R^i(M, N) \xrightarrow{x} \text{Ext}_R^i(M, N) \rightarrow \text{Ext}_R^{i+1}(M/xM, N).$$

For each  $i > r - s$ , we have  $i + 1 > r - s + 1$ , and so the induction hypothesis on  $M/xM$  implies that  $\text{Ext}_R^{i+1}(M/xM, N) = 0$  as  $\text{depth}_R M/xM = s - 1$ . Hence Nakayama's Lemma implies that  $\text{Ext}_R^i(M, N) = 0$  for all  $i > r - s$ . On the other hand the following exact sequence

$$\text{Ext}_R^{r-s}(M, N) \xrightarrow{x} \text{Ext}_R^{r-s}(M, N) \rightarrow \text{Ext}_R^{r-s+1}(M/xM, N) \rightarrow 0$$

and the induction hypothesis on  $M/xM$  imply that  $\text{Ext}_R^{r-s}(M, N) \neq 0$ . □

**Corollary 2.6.** *Let  $(R, \mathfrak{m})$  be a Cohen-Macaulay local ring and let  $N$  be a finitely generated  $R$ -module of finite Gorenstein injective dimension. Then there exists a finitely generated  $R$ -module  $M$  of finite injective dimension such that  $\text{Hom}_R(M, N) \neq 0$  and  $\text{Ext}_R^i(M, N) = 0$  for all  $i > 0$ .*

*Proof.* Let  $\text{depth} R = \dim R = d$ . Then there exist a system of parameter  $x_1, \dots, x_d \in \mathfrak{m} \setminus \mathfrak{m}^2$ . Consider  $R_i = R/(x_1, \dots, x_i)R$  and we know that  $R/(x_1, \dots, x_d)$  is Artinian and injective envelope of  $R/\mathfrak{m}$ , say  $M$ , is a finitely generated  $R_d$ -module. By virtue of [LV, Theorem 3.1], we have  $\text{id}_{R_{d-1}} M = 1$  and repeating this way we have  $\text{id}_R M = d$  and  $M$  is a finitely generated  $R$ -module. We note that by constructing this module and the same reasoning mentioned in [LV, Remarks. p. 319] we have  $\mu_{R_i}^{d-i}(\mathfrak{m}R_i, M) = 1$ . We claim that  $\text{depth}_R M = d$ . Otherwise, assume that  $\text{depth}_R M = t < t+1 = \text{depth}_{R_{d-t-1}} M \leq d$ . As  $R_{d-1}$  is a finitely generated  $R$ -module, by virtue of [LV, Lemma, p. 317], we have  $\text{depth}_{R_{d-t-1}} M = \text{depth}_R M = t$ . On the other hand, Remark 2.3 implies that  $1 = \mu_{R_{d-t-1}}^{t+1}(\mathfrak{m}R_{d-t-1}, M) > \mu_{R_{d-t-1}}^t(\mathfrak{m}R_{d-t-1}, M) > 0$  which is impossible and so  $\text{depth}_R M = d$ . Now, Theorem 2.5 implies that  $\text{Hom}_R(M, N) \neq 0$  and  $\text{Ext}_R^i(M, N) = 0$  for all  $i > 0$ .  $\square$

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